Q1

$f(x)=\left(\frac{x+K}{x-K}\right)^x=\left(\frac{x-K+K+K}{x-K}\right)^x=\left(1+\frac{2K}{x-K}\right)^x$ ricondursi al limite notevole

$\frac{2K}{x-K}=\frac{2}{t}$ -> $t=\frac{x-K}{K}$ -> $x=Kt+K$

$\lim_{x\to+\infty}\left(1+\frac{2K}{x-K}\right)^x=\lim_{t\to+\infty}\left(1+\frac{2}{t}\right)^{Kt+K}=e^{2K}$

$e^{2K}=4$ $\quad 2K=\log_e 4 \Rightarrow K=\frac{1}{2}\log_e 4=\log_e 2$

Q2

$f(x)=3\cos^3 x \qquad P(0,5)$

$3\int\cos^3 x\,dx=3\int\cos^2 x\cos x\,dx=3\int(1-\operatorname{sen}^2 x)\cos x\,dx=$

$=3\int\cos x-\cos x\operatorname{sen}^2 x\,dx=3\int\cos x\,dx-3\int\cos x\operatorname{sen}^2 x\,dx=$

$=3\operatorname{sen}x-\frac{\cancel{3}^{1}}{\cancel{3}}\operatorname{sen}^3 x+c$

Per trovare la primitiva passante per $P(0,5)$ si deve determinare la costante c

$x=0 \quad y=5$

$3\operatorname{sen}0-\operatorname{sen}^3 0+c=5 \quad \to c=5$

Q3

$f(x)=2x^3-3Kx^2+2-K \qquad Df=\forall x\in\mathbb{R}$

$\lim_{x\to+\infty}f(x)=\lim_{x\to+\infty}2x^3=+\infty$

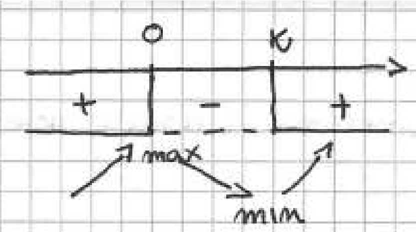
Ha almeno una radice reale

$\lim_{x\to-\infty}f(x)=\lim_{x\to-\infty}2x^3=-\infty$

$D(2x^3-3Kx^2+2-K)=6x^2-6Kx$

$6x^2-6Kx=0 \quad x^2-Kx=0 \quad x(x-K)=0 \quad x_1=0 \quad x_2=K$

$K>0$



$f(0) = 2-K$, con $K>0$ $\qquad f(K) = 2K^3 - 3K^3 + 2 - K = -K^3 + 2 - K = -K^3 - K + 2$

	-1	0	-1	+2
1		-1	-1	-2
	-1	-1	-2	//

$(-K^2 - K - 2)(K-1) = 0$

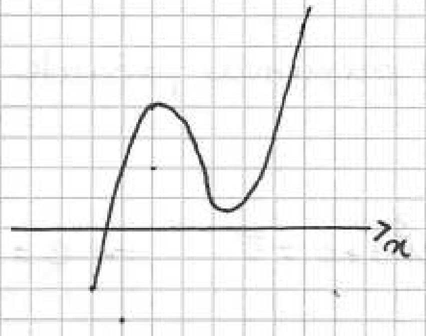
$f(K) \begin{cases} 0 & \text{per } K=1 \\ <0 & \text{per } K>1 \\ >0 & \text{per } K<1 \end{cases}$ $\qquad f(0) \begin{cases} 0 & \text{per } K=2 \\ >0 & \text{per } K<2 \\ <0 & \text{per } K>2 \end{cases}$

Se $0<K<1$

$f(0)>0$

$f(K)>0$

1 soluzione

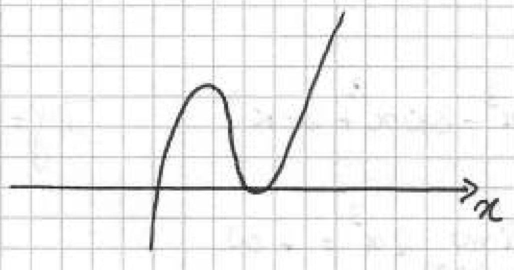


Se $K=1$

$f(K)=0$

$f(0)=1$

3 soluzioni di cui due coincidenti

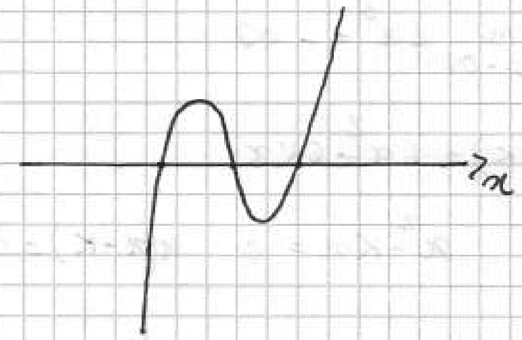


Se $1<K<2$

$f(K)<0$

$f(0)>0$

3 soluzioni distinte

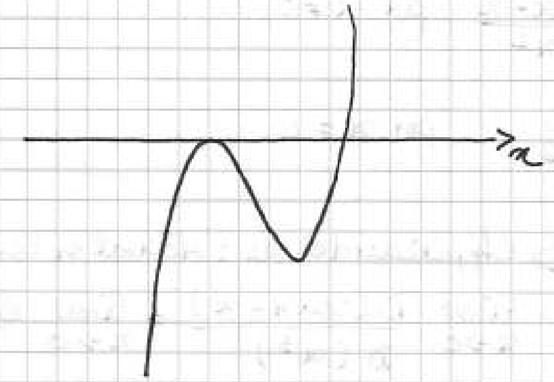


Se $k=2$

$f(0)=0$

$f(k)=-8$

3 soluzioni di cui due coincidenti

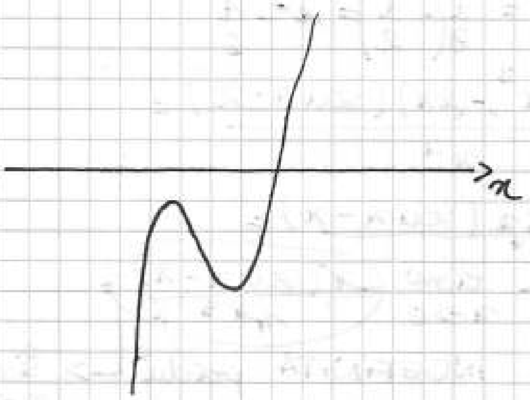


Se $k>2$

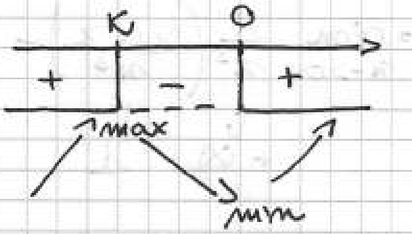
$f(0)<0$

$f(k)<0$

1 soluzione



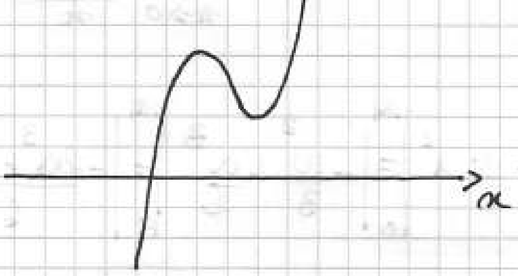
$k<0$



$f(0)=2-k$ con $k<0$ è sempre maggiore di 0 $\forall x\in\mathbb{R}$

$f(k)=(-k^2-k-2)(k-1)$ con $k<0$ è sempre maggiore di 0 $\forall x\in\mathbb{R}$

Quindi se $f(0)>0$ e $f(k)>0$ si ha 1 soluzione



Q4

$$f(x)\begin{cases}\dfrac{\operatorname{sen}x-x}{x^2} & \text{per } x\neq 0\\[2mm] -\dfrac{1}{6} & \text{per } x=0\end{cases}$$

$\lim_{x\to 0}\frac{\operatorname{sen}x-x}{x^2}$ => Applicando de l'Hôpital si ha

$$\lim_{x\to 0}\frac{D(\operatorname{sen}x-x)}{D(x^3)}=\lim_{x\to 0}\frac{\cos x-1}{3x^2}=\lim_{x\to 0}\frac{1}{3}\left(\frac{\cos x-1}{x^2}\right)$$

La funzione è continua in $x=0$ $=\frac{1}{3}\left(-\frac{1}{2}\right)=-\frac{1}{6}$

NB

$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1}{2}$ (limite notevole)

quindi $\lim_{x\to 0}\frac{\cos x-1}{x^2}=-\frac{1}{2}$

$$Df(x)=\frac{(\cos x-1)(x^2)-(2x)(\operatorname{sen}x-x)}{x^4}=$$

$$=\frac{x^2(\cos x-1)-2x(\operatorname{sen}x-x)}{x^4}=$$

$$\lim_{x\to 0}\frac{\cancel{x^2}(\cos x-1)}{\cancel{x^4}\,x^2}-\lim_{x\to 0}\frac{\cancel{2x}(\operatorname{sen}x-x)}{\cancel{x^4}\,x^3}=$$

$\frac{1}{2}$ per il limite notevole

FORMA INDEFINITA perché -> $\lim_{x\to 0}\frac{\operatorname{sen}x}{x^3}-\frac{x}{\cancel{x^3}\,x^2}$

-> $=\lim_{x\to 0}\frac{1}{x^2}\left(\frac{\operatorname{sen}x}{x}\right)-\frac{1}{x^2}$

$\frac{1}{x^2}\to+\infty$, $\frac{\operatorname{sen}x}{x}\to 1$, $-\frac{1}{x^2}\to-\infty$

Applicando de l'Hôpital si ha

$$\lim_{x\to 0}\frac{D(\operatorname{sen}x-x)}{D(x^2)}=\lim_{x\to 0}\frac{\cos x-1}{2x}$$

$$=\lim_{x\to 0}\frac{D(\cos x-1)}{D(2x)}=\lim_{x\to 0}\frac{-\operatorname{sen}x}{2}=0$$

La funzione è derivabile in $x=0$

Q7

$$\lim_{x\to 0}\frac{\int_0^x \operatorname{sen}t^3\,dt}{x^4}=\lim_{x\to 0}\frac{D\int_0^x \operatorname{sen}t^3\,dt}{D\,x^4}=\lim_{x\to 0}\frac{\operatorname{sen}t^3}{4t^3}=\frac{1}{4}$$

-> limite notevole $\lim_{x\to 0}\frac{\operatorname{sen}x}{x}=1$

Q8 $x=ay-y^2$

$$\int_0^{|a|}ay-y^2\,dy=\left|-\frac{1}{3}y^3+\frac{a}{2}y\right|_0^{|a|}=\left|-\frac{1}{3}a^3+\frac{a}{2}a^2\right|_0^{|a|}=\left|-\frac{a^3}{3}+\frac{a^3}{2}\right|_0^{|a|}=\left|\frac{-2a^3+3a^3}{6}\right|_0^{|a|}=\left|\frac{a^3}{6}\right|_0^{|a|}$$

$$\left|\frac{a^3}{6}\right|_0^{|a|}=\frac{32}{6}$$

Se $a>0$ si ha -> $\frac{a^3}{6}=\frac{32}{3}$ -> $a^3=64$ -> $a_1=4$

$a=|4|$

Se $a<0$ si ha -> $-\frac{a^3}{6}=\frac{32}{3}$ -> $-a^3=64$ -> $a_2=-4$

Q10

$f(x) = \text{sen}(\pi x)$

$f(x+T) = f(x)$ con $T = 2$

$f(x+2) = f(x) \rightarrow \text{sen}[\pi(x+2)] = \text{sen}(\pi x)$

$\text{sen}(\pi x + \cancel{2\pi}) = \text{sen}(\pi x)$

→ Dato che il sen x è periodico per 2π non influisce

$\text{sen}(\pi x) = \text{sen}(\pi x)$

QED